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**Option Pricing with Lévy-Stable  
Processes Generated by Lévy-Stable  
Integrated Variance**

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# Option Pricing with Lévy-Stable Processes Generated by Lévy-Stable Integrated Variance

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## Abstract

In this paper we show how to calculate European-style option prices when the log-stock price process follows a Lévy-Stable process with index parameter  $1 \leq \alpha \leq 2$  and skewness parameter  $-1 \leq \beta \leq 1$ . Key to our result is to model integrated variance  $\int_t^T \sigma_s^2 ds$  as an increasing Lévy-Stable process with continuous paths.

Keywords: Lévy-Stable processes, stable Paretian hypothesis, stochastic volatility,  $\alpha$ -stable processes, option pricing, time-changed Brownian motion.

## 1 Introduction

Up until the early 1990's most of the underlying stochastic processes used in the financial literature were based on a combination of Brownian motion and Poisson

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\*We are very grateful for comments from Hu McCulloch and seminar participants at the University of Toronto. Corresponding author: a.carte@bbk.ac.uk

processes. One of the most fundamental assumptions throughout has been that financial asset returns are the cumulative outcome of many small events that happen very frequently at a ‘microscopic level’ in time, so that their impact may be regarded as parameterised continuously by time. If these microscopic events are considered statistically independent with finite variance it is straightforward to characterise their limiting cumulative behaviour, as the timestep tends to zero, by invoking the Central Limit Theorem (CLT). Hence, Gaussian-based distributions are a plausible class of models for financial processes.

More generally, dropping the assumption of finite variance, the sum of many iid events always has, after appropriate scaling and shifting, a limiting distribution termed a Lévy-Stable law; this is the generalised version of the Central Limit Theorem, (GCLT), [ST94]; the Gaussian distribution is one example. Based on this fundamental result, it is plausible to generalise the assumption of Gaussian price increments by modelling the ‘formation’ of prices in the market by the sum of many stochastic events with a Lévy-Stable limiting distribution.

An important property of Lévy-Stable distributions is that of stability under addition: when two independent copies of a Lévy-Stable random variable are added then, up to scaling and shift, the resulting random variable is again Lévy-Stable with the same shape. This property is very desirable in models used in finance and particularly in portfolio analysis and risk management, see for example Fama [Fam71], Ziemba [Zie74] and the more recent work by Tokat and Schwartz [TS02], Ortobelli et al [OHS02] and Mittnik et al [MRS02]. Only for Lévy-Stable distributed returns do we have the property that linear combinations of different return series, for example portfolios, again have a Lévy-Stable distribution [Fel66].

Based on the GCLT we have, in general terms, two ways of modelling stock prices or stock returns. If it is believed that *stock returns* are at least approximately governed by a Lévy-Stable distribution the accumulation of the random events is additive. On the other hand, if it is believed that the *logarithm of stock prices* are approximately governed by a Lévy-Stable distribution then the accumulation is multiplicative. In the literature most models have assumed that log-prices, instead of returns, follow a Lévy-Stable process. McCulloch [McC96] assumes that assets are log Lévy-Stable

and prices options using a utility maximisation argument; more recently Carr and Wu [CW03] priced European options when the log-stock price follows a maximally skewed Lévy-Stable process. Cartea and Howison [CH05] also assume that log prices follow a Lévy-Stable process and provide a solution to the pricing problem as a distinguished limit of the Lévy-Stable process.

Finally, based on Mandelbrot and Taylor [Man97], Platen, Hurst and Rachev [HPR99] provide a model to price European options when returns follow a (symmetric) Lévy-Stable process. In their models the Brownian motion that drives the stochastic shocks to the stock process is subordinated to an intrinsic time process that represents ‘operational time’ on which the market operates. Option pricing can be done within the Black-Scholes framework and one can show that the subordinated Brownian motion is a symmetric Lévy-Stable motion.

The motivation of this paper is as follows. It is well known that if the risk-neutral stock price process follows

$$S_T = S_t e^{r(T-t) - \frac{1}{2} \int_t^T \sigma_s^2 ds + \int_t^T \sigma_s dW_s^Q}, \quad (1)$$

where  $dW_t^Q$  is the increment of the Brownian motion and the volatility is given by a stochastic process  $\sigma_t$  where  $\sigma_t$  and  $W_t^Q$  are independent for all  $0 \leq t \leq T$ , then the value of a European vanilla option written on the underlying stock price  $S_t$  with payoff  $\Pi(S, T)$  is given by

$$V(S, t) = \mathbb{E}^Q \left[ V_{\text{BS}} \left( S_t, t, K, \left( \frac{1}{T-t} \int_t^T \sigma_s^2 ds \right)^{1/2}, T \right) \right], \quad (2)$$

where the expected value is with respect to the random variable  $Y_{t,T} = \int_t^T \sigma_s^2 ds$  under the risk-neutral measure  $Q$  and  $V_{\text{BS}}$  is the usual Black-Scholes value for a European option. In general, the distribution or characteristic function of the integrated variance  $Y_{t,T}$  is not known, so evaluating (2) is not straightforward, although given the characteristic function of the integrated variance we can use standard transform methods to evaluate  $V(S, t)$  given by equation (2). In this paper we propose a two-factor model where the shocks to the stock process are conditionally Gaussian, ie Brownian motion, and the integrated variance  $Y_{t,T}$  follows a Lévy-Stable process, and as a result the distribution of the log-stock prices is Lévy-Stable.

The paper is structured as follows. Section 2 presents definitions and properties of Lévy-Stable processes. In particular we show how symmetric Lévy-Stable random variables may be ‘built’ as a combination of two independent Lévy-Stable random variables. Section 3 discusses the path properties required to model integrated variance as a totally skewed to the right Lévy-Stable process. Section 4 describes the dynamics of the stock process under both the physical and risk-neutral measure and shows how option prices are calculated when the stock returns or log-stock process follows a Lévy-Stable process. Finally, section 5 shows numerical results and section 6 concludes.

## 2 Lévy-Stable random variables

In this section we show how to obtain *any* symmetric Lévy-Stable motion as a stochastic process whose innovations are the product of two independent Lévy-Stable random variables. The only conditions we require (we will make this precise in Proposition 2) are that one of the independent random variables is symmetric and the other is totally skewed to the right. This is a simple, yet very important, result since we can choose a Gaussian random variable as one of the building blocks together with any other totally skewed random variable to ‘produce’ symmetric Lévy-Stable random variables. Furthermore, choosing a Gaussian random variable as one of the building blocks of a symmetric random variable will be very convenient since we will be able to relate any symmetric Lévy-Stable motion as a conditional Brownian motion, conditioned on the other building block, the totally skewed Lévy-Stable random variable which in our case will be the quantity known as integrated variance.

We recall that the log-characteristic function of a Lévy-Stable process  $L_t$  is given by

$$\ln \mathbb{E}[e^{i\theta L_t}] \equiv \Psi_t(\theta) = \begin{cases} -t\kappa^\alpha |\theta|^\alpha \{1 - i\beta \text{sign}(\theta) \tan(\alpha\pi/2)\} + im\theta & \text{for } \alpha \neq 1, \\ -t\kappa |\theta| \{1 + \frac{2i\beta}{\pi} \text{sign}(\theta) \ln |\theta|\} + im\theta & \text{for } \alpha = 1, \end{cases} \quad (3)$$

where the parameter  $\alpha \in (0, 2]$  is known as the stability index;  $\kappa > 0$  is a scaling parameter;  $\beta \in [-1, 1]$  is a skewness parameter and  $m$  is a location parameter. If the

random variable  $L_1$  belongs to a Lévy-Stable distribution with parameters  $\alpha, \kappa, \beta, m$  we write  $L_1 \sim S_\alpha(\kappa, \beta, m)$ . Bearing in mind the translation invariance with respect to  $m$  and the implicit scaling with respect to  $\kappa$  we define a standard Lévy-Stable motion by  $L_t^{\alpha, \beta} \sim S_\alpha(t^{1/\alpha}, \beta, 0)$  and the increment by  $dL_t^{\alpha, \beta}$  is thought of as having the distribution  $S_\alpha(dt^{1/\alpha}, \beta, 0)$ . Finally, we point out that when  $\alpha < 1$  and  $\beta = -1$  (resp.  $\beta = 1$ ) the process  $L_t$  has support on the negative (resp. positive) line.

It is straightforward to see that for the case  $0 < \alpha \leq 1$  the random variable  $L_1$  does not have any moments, and for the case  $1 < \alpha < 2$  only the first moment exists (the case  $\alpha = 2$  is Gaussian). Moreover, given the asymptotic behaviour of the tails of the distribution of a Lévy-Stable random variable it can be shown that the Laplace transform  $\mathbb{E}[e^{-\tau L_1}]$  of  $L_1$  exists only when its distribution is totally skewed to the right, that is  $\beta = 1$ , which we state in the following proposition which we use later.

**Proposition 1. *The Laplace Transform*** [ST94]. *The Laplace transform  $\mathbb{E}[e^{-\tau X}]$  with  $\tau \geq 0$  of the Lévy-Stable variable  $X \sim S_\alpha(\kappa, 1, 0)$  with  $0 < \alpha \leq 2$  and scale parameter  $\kappa > 0$  satisfies*

$$\ln \mathbb{E}[e^{-\tau X}] = \begin{cases} -\kappa^\alpha \tau^\alpha \sec \frac{\pi\alpha}{2} & \text{for } \alpha \neq 1, \\ \frac{2\kappa}{\pi} \tau \ln \tau & \text{for } \alpha = 1. \end{cases} \quad (4)$$

The existence of the Laplace transform of a totally skewed to the right Lévy-Stable random variable will enable us to show how to price options as a weighted average of the classical Black-Scholes price when the shocks to the stock process follow a Lévy-Stable process. First we see that any symmetric Lévy-Stable random variable can be represented as the product of a totally skewed with a symmetric Lévy-Stable variable as shown by the following proposition.

**Proposition 2. *Constructing Symmetric Variables*** [ST94]. *Let  $X \sim S_{\alpha'}(\kappa, 0, 0)$ ,  $Y \sim S_{\alpha/\alpha'}((\cos \frac{\pi\alpha}{2\alpha'})^{\frac{\alpha'}{\alpha}}, 1, 0)$ , with  $0 < \alpha < \alpha' \leq 2$ , be independent. Then the random variable*

$$Z = Y^{1/\alpha'} X \sim S_\alpha(\kappa, 0, 0).$$

Note that we may use Brownian motion as one of the building blocks to obtain symmetric Lévy-Stable processes.

### 3 Stochastic Volatility with Lévy-Stable Shocks

As motivated in the introduction, the Lévy-Stable hypothesis postulates that the shocks to the stock process must be Lévy-Stable. If we assume that the returns process is given by

$$\frac{dS_t}{S_t} = \mu dt + \sigma_t dW_t \quad \text{so that} \quad S_T = e^{\mu(T-t) - \frac{1}{2} \int_t^T \sigma_s^2 ds + \int_t^T \sigma_s dW_s},$$

where  $\mu$  is a constant and  $dW_t$  the increment of Brownian motion we could be tempted, based on Proposition 2, to model volatility by assuming that the integrated variance is given by

$$Y_{t,T} = \int_t^T \sigma_s^2 ds = \int_t^T dL_s^{\alpha/2,1}. \quad (5)$$

Note that  $dL_t^{\alpha/2,1}$  is the increment of a positive Lévy-Stable motion so that (5) is an increasing process. This seems a reasonable choice since

$$\mathbb{E}[e^{i\theta \int_t^T \sigma_s dW_s}] = e^{-\frac{1}{2\alpha/2} \sec(\pi\alpha/4)(T-t)|\theta|^\alpha}$$

hence the shocks to the process would be symmetric Lévy-Stable, see Proposition 2.

Unfortunately this model for integrated variance is inconsistent since on the left-hand side of (5) we have the integrated variance  $\int_t^T \sigma_s^2 ds$  which is, by construction, a continuous process. However, on the right-hand side of the SDE, we have the nonnegative Lévy-Stable motion  $\int_t^T dL_s^{\alpha/2,1}$  which is by construction a purely discontinuous process. The following subsection discusses a way of constructing a process for the integrated variance that is Lévy-Stable but with continuous paths.

#### 3.1 Sample Path Properties: Modelling Integrated Volatility

In this section we show that it is possible to specify a model for stochastic integrated variance whose finite-dimensional distribution is a totally skewed to the right Lévy-Stable distribution possessing continuous paths. We show that a purely discontinuous process such as the Lévy-Stable motion  $\int_t^T dL_s^{\alpha/2,1}$  can be modified to obtain a continuous process by introducing a suitable deterministic function of time  $f(s, T)$  with



$s \in \mathbb{R}^+$  in the kernel of  $\int_t^T f(s, T) dL_s^{\alpha/2, 1}$  to ‘damp’ the jump process and ‘force’ it to be continuous in  $T$ . In fact we will require that  $f(s, T) = 0$  as  $s \rightarrow T$  so the ‘last’ jumps of the process get smoothed out. (For a general discussion of the path behaviour of processes of the type  $\int_t^T f(s, T) dL_s^{\alpha/2, 1}$  see [ST94].) Since we are interested in pricing options where the underlying stochastic component is driven by a symmetric Lévy-Stable process we would like to specify a kernel  $f(s, T)$  so the finite-dimensional distribution of  $\int_t^T \sigma_s^2 ds = \int_t^T f(s, T) dL_s^{\alpha/2, 1}$  is totally skewed to the right Lévy-Stable. As we shall show below, there are many such functions; we denote the class of such functions by  $\mathbb{F}$ . Below we present a proposition that provides sufficient conditions satisfied by the functions in  $\mathbb{F}$ .

**Proposition 3.** *Let  $f(s, T)$  be a continuously differentiable function and define the process  $X_{t,T} = \int_t^T f(s, T) dL_s^{\alpha/2, 1}$ . Then  $X_{t,T}$  is continuous in  $T$ .*

Proof. Using integration by parts we have that

$$\begin{aligned} \int_t^T f(s, T) dL_s^{\alpha/2, 1} &= f(s, T) L_s^{\alpha/2, 1} \Big|_t^T - \int_t^T \frac{\partial f(s, T)}{\partial s} L_s^{\alpha/2, 1} ds \\ &= -f(t, T) L_t^{\alpha/2, 1} - \int_t^T \frac{\partial f(s, T)}{\partial s} L_s^{\alpha/2, 1} ds; \end{aligned}$$

by standard properties of  $L_t^{\alpha/2, 1}$  and since  $f(s, T)$  is continuously differentiable,  $X_{t,T}$  has continuous paths, ie is continuous in  $T$ . ■

Two possible choices for  $f(s, T)$  are

$$f(s, T) = g(T - s) = T - s \quad T \geq s \geq 0, \quad (6)$$

$$f(s, T) = g(T - s) = \frac{1}{\gamma} (1 - e^{-\gamma(T-s)^n}) \quad \text{for } T, s \geq 0 \text{ and } n \geq 1, \quad (7)$$

where  $\gamma$  is a positive constant that can be seen as a damping factor which we can choose freely; when  $n = 1$  we get an Ornstein-Uhlenbeck-type process (OU-type), an ‘extension’ of an OU process in which instead of the shocks being driven by Brownian

motion they are driven by a Lévy process, see [Wol82]. Barndorff-Nielsen and Shephard [BNS02] were the first to introduce OU-type stochastic volatility models driven by positive Lévy processes. A third choice is

$$f(s, T) = g(T - s) = \ln(T - s + 1) \quad \text{for} \quad T \geq s \geq 0. \quad (8)$$

Note that for some purposes it is convenient to require that  $f(s, T) \geq 0$ ; all the examples above have this property.

### 3.2 Illustration

We now illustrate the different building blocks needed to obtain the integrated variance process described above. First we simulate a totally skewed to the right Lévy-Stable motion; then we get the spot variance process, by choosing an appropriate kernel; then we produce the integrated variance process. We focus on kernels of the integrated variance of the form

$$f(s, T) = g(T - s) = \frac{1}{\gamma} (1 - e^{-\gamma(T-s)^n}).$$

The solid line in the two bottom graphs of Figure 1 represents the case with  $n = 1$ ,  $t = 0$ ,  $0 \leq T \leq 1$  and  $\gamma = 25$  which would yield a standard OU-type process. In the same figure the dotted lines represent the case  $n = 1.2$ ,  $T = 1$  and  $\gamma = 25$ . Note that the higher the constant  $n$  is the ‘smoother’ is the path of the integrated variance.

## 4 Model dynamics and option prices

In this section we present the model dynamics for the stock price and show how to price vanilla options. For ease of presentation subsection 4.1 looks at a model where the shocks to the returns or log-stock process are symmetric and then subsection 4.2 extends it to a model where shocks can also be asymmetric. Finally, subsection 4.3 shows how to price vanilla options when the shocks to the underlying stock process follow a Lévy-Stable process for  $\alpha > 1$  and  $-1 \leq \beta \leq 1$ .

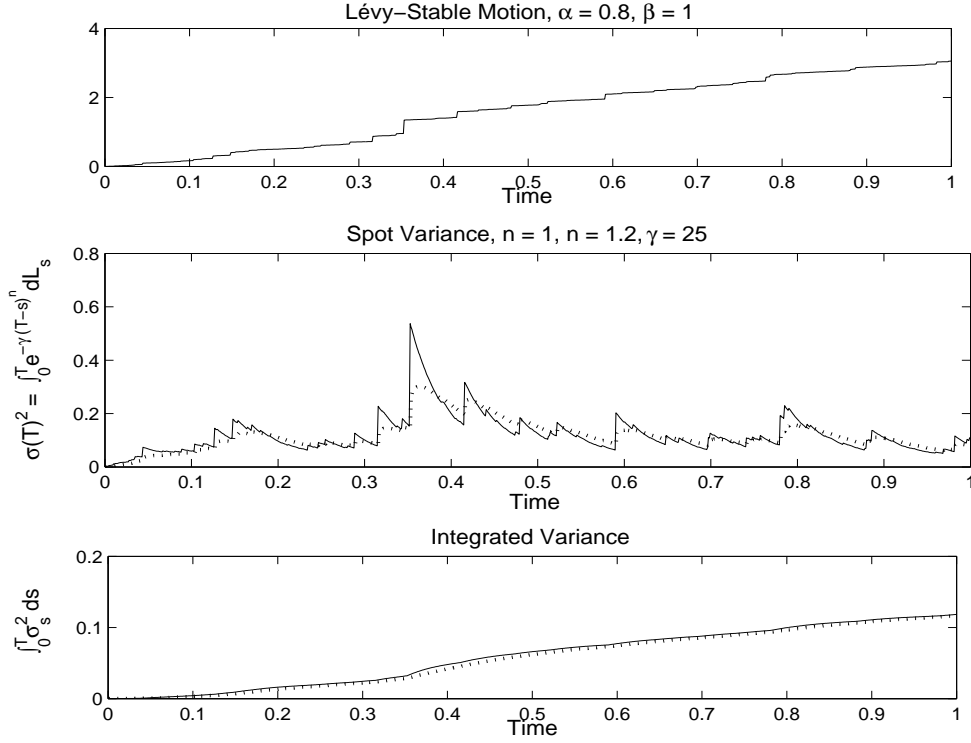


Figure 1: Simulated integrated variance with kernel  $g(T-s) = 25^{-1} (1 - e^{-25(T-s)^n})$  with  $n = 1$ ,  $T = 1$ , solid line, and  $n = 1.2$ ,  $T = 1$ , dotted line.

Given the nature of the model it is obvious that there will not be a unique equivalent martingale measure (EMM). In line with most of the Lévy process literature we choose an EMM that is structure preserving since, among other features (see [CT04]), transform methods for pricing are straightforward to implement; this will be discussed at the end of subsection 4.2.

## 4.1 Modelling returns

As pointed out in the introduction we can either model returns or stock prices. In our case we may assume that when shocks are symmetric we can take either route. For example, if we believe that the shocks to the returns process follow a Lévy-Stable

distribution we assume that

$$\frac{dS_t}{S_t} = \mu dt + \sigma_t dW_t \quad (9)$$

$$\int_t^T \sigma_s^2 ds = \hat{\sigma}^{\alpha/2} \int_t^T g(T-s) dL_s^{\alpha/2,1}, \quad (10)$$

where  $dW_t$  denotes the increment of the standard Brownian motion,  $g(T-s) \in \mathbb{F}$ ,  $\hat{\sigma} \geq 0$  and  $\mu$  are constants. In appendix A we show that by modelling integrated variance as in (10) the shocks to the stock process (9) are symmetric Lévy-Stable.

Note that we might also stipulate that our departure point is the risk-neutral dynamics for the stock process and that our model is given as above with  $\mu = r$ . In this case the risk-neutral dynamics follow

$$\frac{dS_t}{S_t} = r dt + \sigma_t dW_t^Q \quad (11)$$

with  $\int_t^T \sigma_s^2 ds$  as in (10). However, we need not specify the risk-neutral dynamics as a starting point since it is possible to postulate the physical dynamics and then choose an EMM. We discuss this change of measure below for the model that also allows for asymmetric Lévy-Stable shocks and the symmetric case then becomes a particular case.

Before proceeding we remark that the stochastic integral  $\int_t^T \sigma_s dW_s$  can be seen as a time-changed Brownian motion [KS02]. In this case the integrated variance  $\int_t^T \sigma_s^2 ds$  represents the time-change and it is straightforward to show that

$$\int_t^T \sigma_s dW_s \stackrel{d}{=} W_{\hat{T}_{t,T}}$$

where  $\hat{T}_{t,T} = \int_t^T \sigma_s^2 ds$ .

## 4.2 Modelling Log-Stock Prices

Financial data suggests that returns are skewed rather than symmetric, see for example [KL76], [CLM97], [CW03]. The symmetric model above can be extended to allow the dynamics of the log-stock process to follow an asymmetric Lévy-Stable process.

In stochastic volatility models one way to introduce skewness in the log-stock process is to correlate the random shocks of the volatility process to the shocks of the stock process. It is typical in the literature to assume that the Brownian motion of the stock process, say  $dW_t$ , is correlated with the Brownian motion of the volatility process, say  $dZ_t$ . Thus  $\mathbb{E}[dW_t dZ_t] = \rho dt$  and we can write  $\tilde{Z}_t = \rho W_t + \sqrt{1 - \rho^2} Z_t$ , where  $\tilde{Z}_t$  is independent of  $W_t$ . The correlation parameter  $\rho$  is also known in the literature as the leverage effect and empirical studies suggest that  $\rho < 0$  [FPS00]. In our case we may also include a leverage effect via a parameter  $\ell$  to produce skewness in the stock returns. However, the notion of ‘correlation’ does not apply in our case because for Lévy-Stable random variables, as given that moments of second and higher order do not exist, nor do correlations.

Hence to allow for asymmetric Lévy-Stable shocks, under the physical measure we assume that

$$\ln(S_T/S_t) = \mu(T-t) + \int_t^T \sigma_s dW_s + \ell \tilde{\sigma}^\alpha \int_t^T d\tilde{L}_s^{\alpha,-1} \quad (12)$$

$$\int_t^T \sigma_s^2 ds = \hat{\sigma}^{\alpha/2} \int_t^T g(T-s) dL_s^{\alpha/2,1}. \quad (13)$$

Here  $dW_t$  denotes the increment of the standard Brownian motion independent of both  $d\tilde{L}_t^{\alpha,-1}$  and  $dL_t^{\alpha/2,1}$  and we note that  $d\tilde{L}_t^{\alpha,-1}$  is totally skewed to the left and that  $\alpha < 2$ , ie the stability index  $\alpha$  is not restricted to be less than unity. Moreover,  $\mu, \tilde{\sigma} \geq 0, \hat{\sigma} \geq 0$  are constants,  $g(T-s) \in \mathbb{F}$  and the leverage parameter  $\ell \geq 0$ .<sup>1</sup> In appendix B we show that the shocks to the price process are asymmetric Lévy-Stable.

Before proceeding we discuss the connection of the dynamics of the stock price under the physical measure  $P$  and the risk-neutral measure  $Q$ . Recall that a probability

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<sup>1</sup>Note that here we model log-stock prices since we *cannot* include a leverage effect in equation (9) in the form

$$\begin{aligned} \frac{dS_t}{S_t} &= \mu dt + \sigma_t dW_t + \ell \tilde{\sigma}^\alpha d\tilde{L}_t^{\alpha,-1} \\ \int_t^T \sigma_s^2 ds &= \hat{\sigma}^{\alpha/2} \int_t^T g(T-s) dL_s^{\alpha/2,1}, \end{aligned} \quad (14)$$

because the solution to the SDE with leverage (14) will deliver a stock process  $S_t$  that allows negative prices due to the jumps of the increments of the Lévy-Stable motion  $d\tilde{L}_t^{\alpha,-1}$ .

measure  $Q$  is called an EMM if it is equivalent to the physical probability  $P$  and the discounted price process is a martingale. It is straightforward to see that in the model proposed here the set of EEMs is not unique, hence we must motivate the choice of a particular EMM. Based on Theorem 3.1 in [NV03] we choose a structure-preserving measure where the risk-neutral dynamics of the model (12) and (13) follows

$$\ln(S_T/S_t) = r(T-t) - \int_t^T \sigma_s^2 ds + \frac{1}{2}(T-t)\ell^\alpha \sigma_\ell^\alpha \sec \frac{\pi\alpha}{2} + \int_t^T \sigma_s dW_s^Q + \ell\tilde{\sigma}^\alpha \int_t^T d\tilde{L}_s^{\alpha,-1}.$$

Note that if  $\ell = 0$  we obtain the risk-neutral dynamics for the case when the returns or log-stock process follows a symmetric Lévy-Stable process under  $P$ .

### 4.3 Option Pricing with Lévy-Stable Volatility

The preceding sections were devoted to finding a suitable model for stochastic volatility that would enable us to model the unconditional returns process or log-stock process as a Lévy-Stable process. Moreover, as motivated in the introduction by equations (1) and (2), it is straightforward to see that if we assume the dynamics given by (12) and (13) the price of a vanilla option is given by the iterated expectations

$$V(S, t) = \mathbb{E}_{\tilde{L}_t^{\alpha,-1}}^Q \left[ \mathbb{E}_{\sigma_t}^Q \left[ \mathbb{E}^Q \left[ V_{BS} \left( S_t e^{\ell \int_t^T d\tilde{L}_s}, t, K, \left( \frac{1}{T-t} \int_t^T \sigma_s^2 ds \right)^{1/2}, T \right) \middle| \tilde{L}_t^{\alpha,-1}, \sigma_t \middle| \tilde{L}_t^{\alpha,-1} \right] \right] \right], \quad (15)$$

where  $Q$  is the risk-neutral measure and  $V_{BS}$  is the Black-Scholes value for a European option.

**Remark 1.** Note that if we let  $g(T-s) = 0$  then the model reduces to

$$\ln(S_T/S_t) = \mu(T-t) + \ell\tilde{\sigma}^\alpha \int_t^T d\tilde{L}_s^{\alpha,-1},$$

which is the Finite Moment Log-Stable (FMLS) model of [CW03].

**Proposition 4.** It is possible to extend the results above to price European call and put options when the skewness coefficient  $\beta \in [0, 1]$ .

**Proof.** Using put-call inversion [McC96], we have by no-arbitrage that European call and put options are related by

$$C(S, t; K, T, \alpha, \beta) = SKP(S^{-1}, t; K^{-1}, T, \alpha, -\beta).$$

■

Note that using put-call inversion allows us to obtain put prices when the log-stock price follows a positively skewed Lévy-Stable process, based on call prices where the underlying log-stock price follows a negatively skewed Lévy-Stable process. Furthermore, put-call parity allows us to obtain call prices when the skewness parameter  $-1 \leq \beta \leq 0$ .

As an example, we can use the approach above to derive closed-form solutions for option prices when the random shocks to the price process are distributed according to a Cauchy Lévy-Stable process,  $\alpha = 1$  and  $\beta = 0$ .

**Remark 2. Closed-form Solution when Returns follow a Cauchy Process.** By letting  $\alpha = 1$  and  $\ell = 0$  in (12) and (13) we have that option prices, under the risk-adjusted measure  $Q$ , are given by

$$V(S, t) = \frac{\int_t^T g(T-s)^{1/2} ds}{(T-t)\sqrt{2\pi}} \int_0^\infty V_{BS}(S_t, t, K, \bar{Y}_{t,T}^{1/2}, T) \frac{1}{\bar{y}^{3/2}} e^{-\left(\frac{\int_t^T g(T-s)^{1/2} ds}{T-t}\right)^2 / 2\bar{y}} d\bar{y},$$

where  $\bar{Y}_{t,T} = \frac{1}{T-t} \int_t^T \sigma_s^2 ds$ .

To see this, first we note that the combination of a Gaussian, the Brownian motion in (12), and Lévy-Smirnov  $S_{1/2}(\kappa, 1, 0)$ , the process followed by the integrated variance in (13), random variables results in a Cauchy random variable  $S_1(\kappa, 0, 0)$ . This can be seen by calculating the convolution of their respective pdf's. Now, recall that the pdf for a Lévy-Smirnov random variable  $S_{1/2}(\kappa, 1, 0)$  is given by  $(\kappa/2\pi)^{1/2} x^{-3/2} e^{-\kappa/2x}$  with support  $(0, \infty)$ ; hence in our case the distribution of the average integrated variance is given by

$$\bar{Y} \equiv \frac{1}{T-t} \int_t^T g(T-s) dL_s^{\alpha/2, 1} \sim S_{1/2} \left( \frac{1}{(T-t)^2} \left( \int_t^T g(T-s)^{1/2} ds \right)^2, 1, 0 \right);$$

thus the value of the option is

$$V(S, t) = \frac{\int_t^T g(T-s)^{1/2} ds}{(T-t)\sqrt{2\pi}} \int_0^\infty V_{BS}(S_t, K, \bar{Y}_{t,T}^{1/2}, T) \frac{1}{\bar{y}^{3/2}} e^{-\left(\frac{\int_t^T g(T-s)^{1/2} ds}{T-t}\right)^2 / 2\bar{y}} d\bar{y}.$$

## 5 Numerical illustration: Lévy-Stable Option Prices

In this section we show how vanilla option prices are calculated according to the above derivations. One route is to calculate the expected value of the Black-Scholes formula weighted by the stochastic volatility component and the leverage effect. Another route to price vanilla options for stock prices that follow a geometric Lévy-Stable processes is to compute the option value as an integral in Fourier space, using Complex Fourier Transform techniques [Lew01], [CM99].

We use the Black-Scholes model as a benchmark to compare the option prices obtained when the returns follow a Lévy-Stable process. Our results are consistent with the findings in [HW87] where the Black-Scholes model underprices in- and out-of-the-money call option prices and overprices at-the-money options.

### 5.1 Option Prices for Symmetric Lévy-Stable log-Stock Prices

In this subsection we obtain option prices and implied volatilities when the log-stock prices follow symmetric Lévy-Stable process. Recall that, under the risk-neutral measure  $Q$ , the stock price and variance process are given by

$$\begin{aligned} S_T &= S_t e^{r(T-t) - \frac{1}{2} \int_t^T \sigma_s^2 ds + \int_t^T \sigma_s dW_s^Q}, \\ \int_t^T \sigma_s^2 ds &= \hat{\sigma}^{\alpha/2} \int_t^T g(T-s) dL_s^{\alpha/2, 1}. \end{aligned}$$

The first step we take is to calculate the characteristic function of the process

$$Z_{t,T} = -\frac{1}{2} \int_t^T \sigma_s^2 ds + \int_t^T \sigma_s dW_s^Q.$$



**Proposition 5.** *The characteristic function of  $Z_{t,T}$  is given by*

$$\mathbb{E}^Q[e^{i\xi Z_{t,T}}] = e^{-\frac{1}{2}\sigma_{LS}^\alpha (i\xi + \xi^2)^{\alpha/2} \int_t^T g(T-s)^{2/\alpha} ds}, \quad (16)$$

where  $\xi = \xi_r + i\xi_i$  and  $-1 \leq \xi_i \leq 0$  and  $\sigma_{LS} \geq 0$  (see (20) in appendix A). Moreover, the characteristic function is analytic in the strip  $-1 < \xi_i < 0$ .

**Proof.** The characteristic function is given by

$$\begin{aligned} \mathbb{E}^Q[e^{i\xi Z_{t,T}}] &= \mathbb{E}^Q\left[e^{-\frac{1}{2}i\xi \int_t^T \sigma_s^2 ds + i\xi \int_t^T \sigma_s dW_s}\right] \\ &= \mathbb{E}^Q\left[e^{-\frac{1}{2}i\xi \int_t^T \sigma_s^2 ds - \frac{1}{2}\xi^2 \int_t^T \sigma_s^2 ds}\right] \\ &= \mathbb{E}^Q\left[e^{-\frac{1}{2}(i\xi + \xi^2) \int_t^T g(T-s) dL_s^{\alpha/2,1}}\right] \\ &= e^{-\frac{1}{2}\sigma_{LS}^\alpha (i\xi + \xi^2)^{\alpha/2} \int_t^T g(T-s)^{2/\alpha} ds}. \end{aligned}$$

The last step is possible since the expected value exists if  $\xi$  is restricted so that  $\xi_r^2 - \xi_i^2 + \xi_i \geq 0$ , by consideration of the penultimate line. The region where this is true contains the strip  $-1 \leq \xi_i \leq 0$ . Finally, it is straightforward to observe that the characterisitic function is analytic in this strip. ■

To price call options we proceed as above and use the following expression:

$$C(x, t) = e^{x_t} - \frac{1}{2\pi} e^{-r(T-t)} K \int_{i\xi_i - \infty}^{i\xi_i + \infty} e^{-i\xi x_t} \frac{K^{i\xi}}{\xi^2 - i\xi} e^{(T-t)\Psi(-\xi)} d\xi \quad (17)$$

where  $x_t = \ln S_t$ ,  $0 < \xi_i < 1$ , and  $\Psi(\xi)$  is the characteristic function of the process  $\ln S_T$ .

### 5.1.1 Numerics for Symmetric Lévy-Stable log-Stock Prices

We now calculate European-style option prices when log-stock or stock returns are symmetric Lévy-Stable using (17). In order to compare these prices with those obtained using the Black-Scholes pricing formula, we have to decide how to choose the

relevant parameters of the two models. In fact, the only parameter that we must carefully examine is the scaling parameter of the Lévy-Stable process; we opt for one that can be related to the standard deviation used when the classical Black-Scholes model is used. One approach is to proceed as in [HPR99] and match a given percentile of the Normal and a symmetric Lévy-Stable distribution. For example, if we want to match the first and third quartile of a Brownian motion with standard deviation  $\sigma = 0.20$  to a symmetric Lévy-Stable motion  $\kappa dL^{\alpha,0}$  with characteristic exponent  $\alpha = 1.7$ , we would require the scaling parameter  $\kappa = 0.1401$ . We have chosen these parameters so that for options with 3 months to expiry these quartiles match. Moreover, in the examples below, we use the kernel  $g(T-s) = \frac{1}{25} (1 - e^{-25(T-s)})$  where for illustrative purposes we have assumed mean-reversion over a two week period, ie  $\gamma = 25$ .

Figure 2 shows the difference between European call options when the stock returns are distributed according to a symmetric Lévy-Stable motion with  $\alpha = 1.7$  and when returns follow a Brownian motion with annual volatility  $\sigma_{BS} = 0.20$ . The figure shows that for out-of-the-money call options the Lévy-Stable call prices are higher than the Black-Scholes and for at-the-money options Black-Scholes delivers higher prices. These results are a direct consequence of the heavier tails under the Lévy-Stable case.

## 5.2 Option Prices for Asymmetric Lévy-Stable log-Stock Prices

In this subsection we obtain option prices and implied volatilities when there is a negative leverage effect, ie log-stock prices follow an asymmetric Lévy-Stable process. Recall that, under the risk-neutral measure  $Q$ , the stock price and variance process are given by

$$\begin{aligned} S_T &= S_t e^{r(T-t) - \frac{1}{2} \int_t^T \sigma_s^2 ds + \frac{1}{2} (T-t) \ell^\alpha \sigma_\ell^\alpha \sec \frac{\pi\alpha}{2} + \int_t^T \sigma_s dW_s^Q + \ell \tilde{\sigma}^\alpha \int_t^T d\tilde{L}_s^{\alpha,-1}}, \\ \int_t^T \sigma_s^2 ds &= \hat{\sigma}^{\alpha/2} \int_t^T g(T-s) dL_s^{\alpha/2,1}. \end{aligned}$$

We proceed as above and calculate the characteristic function of the process

$$Z_{t,T}^\ell = -\frac{1}{2} \int_t^T \sigma_s^2 ds + \int_t^T \sigma_s dW_s + \ell \tilde{\sigma}^\alpha \int_t^T d\tilde{L}_s^{\alpha,-1}.$$

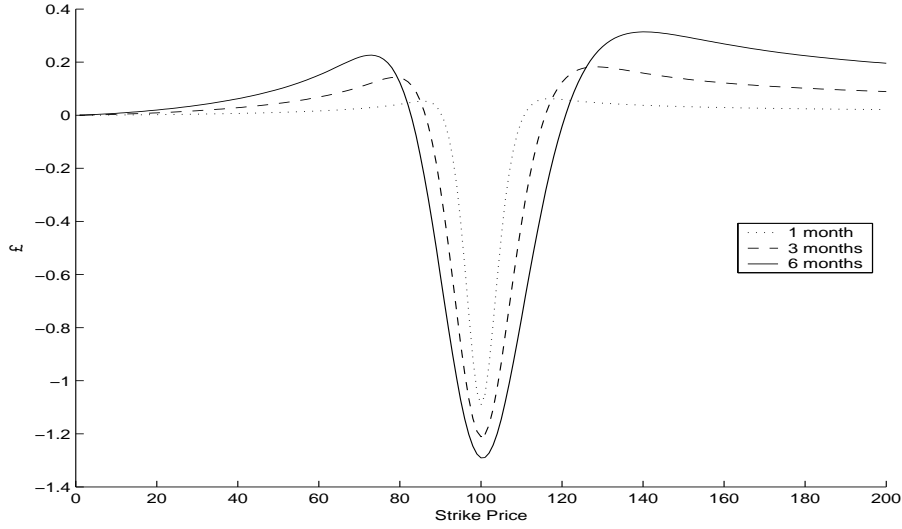


Figure 2: Difference between Lévy-Stable and Black-Scholes call option prices for different expiry dates: one, three and six months. In the Black-Scholes annual volatility is  $\sigma_{BS} = 20\%$ .

**Proposition 6.** *The characteristic function of  $Z_{t,T}^\ell$  is given by*

$$\mathbb{E}^Q[e^{Z_{t,T}^\ell}] = e^{-\frac{1}{2} \left[ \sigma_{LS}^\alpha (i\xi + \xi^2)^{\alpha/2} \int_t^T g(T-s)^{2/\alpha} ds + (T-t)(i\xi\ell)^\alpha \sigma_\ell^\alpha \sec \frac{\pi\alpha}{2} \right]}, \quad (18)$$

where  $-1 \leq \xi_i \leq 0$ ,  $\xi = \xi_r + i\xi_r$ . Moreover, the characteristic function is analytic in the strip  $-1 < \xi_i < 0$ .

**Proof.** The proof is very similar to the one above. It suffices to note that for  $\xi_i \leq 0$

$$\begin{aligned} \left| \mathbb{E}^Q \left[ e^{i\xi \int_t^T d\tilde{L}_s^{\alpha,-1}} \right] \right| &\leq \mathbb{E}^Q \left[ \left| e^{i\xi \int_t^T d\tilde{L}_s^{\alpha,-1}} \right| \right] \\ &= \mathbb{E}^Q \left[ e^{-\xi_i \int_t^T d\tilde{L}_s^{\alpha,-1}} \right] \\ &< \infty. \end{aligned}$$

Moreover, for  $\xi_i < 0$  we have that  $\mathbb{E}^Q \left[ e^{i\xi \int_t^T d\tilde{L}_s^{\alpha,-1}} \right]$  is analytic, ie

$$\begin{aligned} \left| \frac{d}{d\xi} \mathbb{E}^Q \left[ e^{i\xi \int_t^T d\tilde{L}_s^{\alpha,-1}} \right] \right| &= \left| \mathbb{E}^Q \left[ i \int_t^T d\tilde{L}_s^{\alpha,-1} e^{i\xi \int_t^T d\tilde{L}_s^{\alpha,-1}} \right] \right| \\ &< \infty. \end{aligned}$$

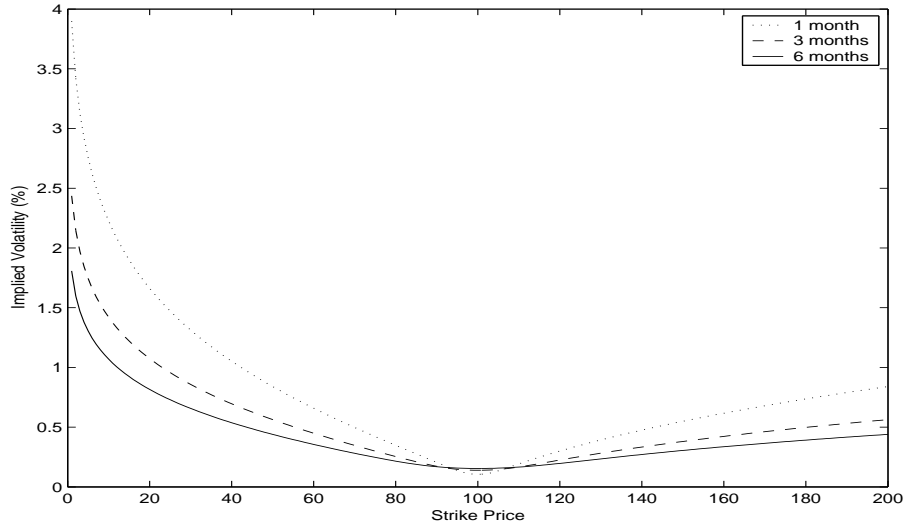


Figure 3: Black-Scholes implied volatility for the Lévy-Stable call option prices when returns follow a symmetric Lévy-Stable motion with  $\alpha = 1.7$ ,  $\beta = 0$  and three expiry dates: one, three and six months.

Putting these results together with the results from Proposition 5 we get the desired result. Note that the requirement is  $-1 < \xi_i < 0$  because  $d\tilde{L}_t^{\alpha, -1}$  is totally skewed to the left, therefore we need  $-\xi_i > 0$ .

■

We use the same  $g(T - s)$  as above and include a leverage parameter  $\ell = 1$  so that returns follow a negatively skewed process with  $\beta(t, T) = -0.5$  when there is 3 months to expiry. Figure 4 shows the difference between Lévy-Stable and Black-Scholes call option prices for different expiry dates. In the Black-Scholes case annual volatility is  $\sigma_{BS} = 0.20$  and in the asymmetric Lévy-Stable case with scaling coefficient  $\sigma_\ell = 0.1401$  (see (20) and (21) in appendix B). Finally, Figure 5 shows the corresponding implied volatility. The negative skewness introduced produces a ‘hump’ for call prices with strike below 100. This is financially intuitive since relative to the Black-Scholes the risk-neutral probability of the call option ending out-of-the-money is substantially higher in the Lévy-Stable case.

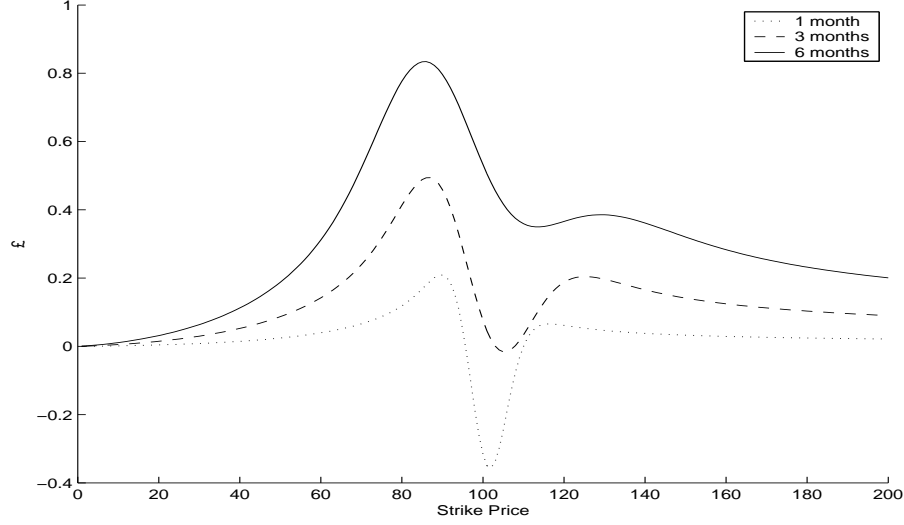


Figure 4: Difference between Lévy-Stable and Black-Scholes call option prices for different expiry dates: one, three and six months. In the Black-Scholes annual volatility is  $\sigma_{BS} = 0.20$  and in the asymmetric Lévy-Stable case the scaling parameters are  $\sigma_{LS} = 0.7673$  and  $\sigma_{\ell} = 0.1401$ .

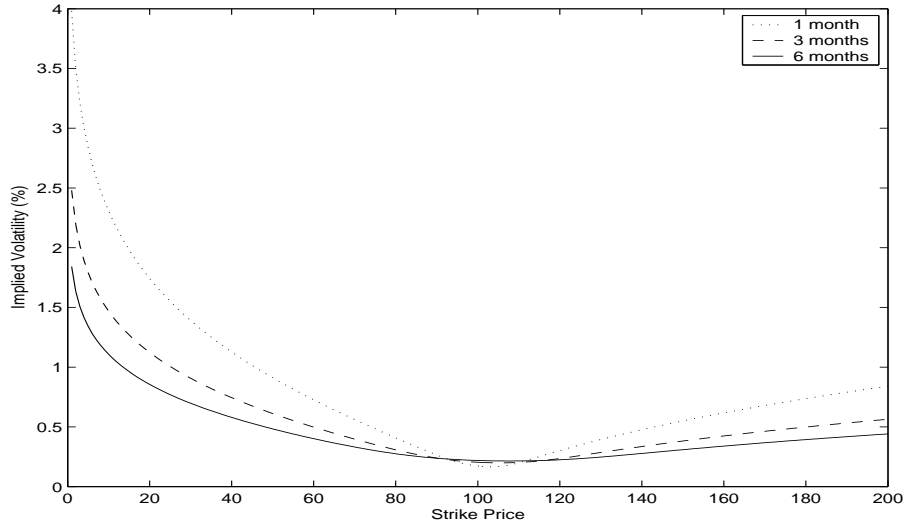


Figure 5: Black-Scholes implied volatility for the Lévy-Stable call option prices when returns follow a symmetric Lévy-Stable motion with  $\alpha = 1.7$ ,  $\sigma_{LS} = 0.7673$  and  $\sigma_{\ell} = 0.1401$  and three expiry dates: one, three and six months.

## 6 Conclusion

The GCLT provides a very strong theoretical foundation to argue that the limiting distribution of stock returns or log-stock prices follow a Lévy-Stable process. In this paper we have shown that it is possible to model stock returns and log-stock prices where the stochastic component is Lévy-Stable distributed covering the whole range of skewness  $\beta \in [-1, 1]$ . We showed that European-style option prices are straightforward to calculate using transform methods and we compare them to Black-Scholes prices where we obtain the expected volatility smile encountered in the markets. Moreover, we show that we can model integrated variance directly as an increasing continuous Lévy-Stable process.

## A Appendix A

Here we show that if the stock process, as assumed above in section 4.1, follows

$$\begin{aligned}\frac{dS_t}{S_t} &= \mu dt + \sigma_t dW_t \\ \int_t^T \sigma_s^2 ds &= \hat{\sigma}^{\alpha/2} \int_t^T g(T-s) dL_s^{\alpha/2,1},\end{aligned}$$

where  $dW_t$  denotes the increment of the standard Brownian motion,  $g(T-s) \in \mathbb{F}$ ,  $\hat{\sigma}$  and  $\mu$  are constants, it is straightforward to show that the shocks to the process are symmetric Lévy-Stable.

First note that the stochastic component of the log-stock process is given by

$$U_{t,T} = \int_t^T \sigma_s dW_s. \quad (19)$$

and for convenience choose<sup>2</sup>

$$\hat{\sigma} = 2 \left( \frac{1}{2} \cos \frac{\pi\alpha}{4} \right)^{2/\alpha} \sigma_{LS}^2. \quad (20)$$

Now we calculate the characteristic function of the random process  $U_{t,T}$ . We have

$$\mathbb{E}[e^{i\theta U_{t,T}}] = \mathbb{E}[e^{i\theta \int_t^T \sigma_s dW_s}],$$

and conditioning on the path of  $\sigma_s$  for  $t \leq s \leq T$  and using iterated expectations we get

$$\mathbb{E}[e^{i\theta U_{t,T}}] = \mathbb{E} \left[ e^{-\frac{1}{2}\theta^2 \int_t^T \sigma_s^2 ds} \right].$$

Now, given that  $\int_t^T \sigma_s^2 ds = \int_t^T g(T-s) dL_s^{\alpha/2,1}$  and using Proposition 1 we write

$$\begin{aligned}\mathbb{E}[e^{i\theta U_{t,T}}] &= \mathbb{E} \left[ e^{-\frac{1}{2}\theta^2 \int_t^T g(T-s) dL_s^{\alpha/2,1}} \right] \\ &= e^{-\frac{1}{2}\sigma_{LS}^\alpha \int_t^T g(T-s)^{2/\alpha} ds |\theta|^\alpha}.\end{aligned}$$

This is clearly the characteristic function of a symmetric Lévy-Stable process with index  $\alpha$ .

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<sup>2</sup>We chose  $\hat{\sigma}$  in this way just for convenience in the calculations since it does not have any effect on the overall qualitative result that the shocks in the process are symmetric Lévy-Stable.

## B Appendix B

Suppose that the stock process, as assumed above in section 4.2, follows

$$\begin{aligned}\ln(S_T/S_t) &= \mu(T-t) + \int_t^T \sigma_s dW_s + \ell \tilde{\sigma}^\alpha \int_t^T d\tilde{L}_s^{\alpha,-1} \\ \int_t^T \sigma_s^2 ds &= \hat{\sigma}^{\alpha/2} \int_t^T g(T-s) dL_s^{\alpha/2,1},\end{aligned}$$

under  $P$  where  $dW_t$  denotes the increment of the standard Brownian motion independent of both  $d\tilde{L}_t^{\alpha,-1}$  and  $dL_t^{\alpha/2,1}$ . Then it is straightforward to verify that the shocks to the above log-stock process under the measure  $P$  are those of a Lévy-Stable process with negative skewness  $\beta \in (-1, 0]$ . Let  $G(t, T) = \int_t^T g(T-s)^{\alpha/2} ds$  and, for simplicity in the calculations, assume that  $\hat{\sigma}$  is given by (20) and

$$\tilde{\sigma} = \frac{1}{2^{1/\alpha}} \sigma_\ell. \quad (21)$$

Now consider the process

$$U_{t,T}^\ell = \int_t^T \sigma_s dW_s + \ell \int_t^T d\tilde{L}_s^{\alpha,-1}.$$

The characteristic function of  $U_{t,T}^\ell$  is given by

$$\begin{aligned}\mathbb{E} \left[ e^{i\theta U_{t,T}^\ell} \right] &= \mathbb{E} \left[ e^{i\theta \left( \int_t^T \sigma_s dW_s + \ell \tilde{\sigma}^\alpha \int_t^T d\tilde{L}_s^{\alpha,-1} \right)} \right] \\ &= e^{-\frac{1}{2} G(t,T) \sigma_{LS}^\alpha |\theta|^\alpha} \mathbb{E} \left[ e^{i\theta \ell \tilde{\sigma}^\alpha \int_t^T d\tilde{L}_s^{\alpha,-1}} \right] \\ &= e^{-\frac{1}{2} G(t,T) \sigma_{LS}^\alpha |\theta|^\alpha} e^{-\frac{1}{2} (T-t) \ell^\alpha \sigma_\ell^\alpha |\theta|^\alpha \{1 + i \operatorname{sign}(\theta) \tan(\pi\alpha/2)\}} \\ &= e^{-\frac{1}{2} (G(t,T) \sigma_{LS}^\alpha + (T-t) \ell^\alpha \sigma_\ell^\alpha) |\theta|^\alpha \left\{ 1 - \frac{-(T-t) \ell^\alpha \sigma_\ell^\alpha}{G(t,T) \sigma_{LS}^\alpha + (T-t) \ell^\alpha \sigma_\ell^\alpha} i \operatorname{sign}(\theta) \tan(\pi\alpha/2) \right\}}.\end{aligned}$$

This is obviously the characteristic function of a skewed Lévy-Stable process with skewness parameter

$$\beta(t, T) = \frac{-(T-t) \ell^\alpha \sigma_\ell^\alpha}{G(t, T) \sigma_{LS}^\alpha + (T-t) \ell^\alpha \sigma_\ell^\alpha} \in (-1, 0].$$

Moreover, when  $\ell = 0$  we obtain  $\beta = 0$  and  $\beta \rightarrow -1$  as  $\ell \rightarrow \infty$ .

Note that the integrated variance does not have a finite first moment since  $\alpha/2 < 1$ . However, in the case of the leverage effect  $\int_t^T d\tilde{L}_s^{\alpha,-1}$  its first moment exists, ie  $\mathbb{E}[\int_t^T d\tilde{L}_s^{\alpha,-1}] < \infty$  since  $1 < \alpha < 2$ .



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